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Inequalities of the Hermite-Hadamard Type for Stochastic Processes with Convexity-Preserving Properties

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Abstract

In this study, we delve into the exploration of Hermite-Hadamard type inequalities specifically tailored for stochastic processes with convexity-preserving properties. The primary research objective is to examine the convexity characteristics of the derivatives of these processes, establishing new bounds and providing deeper insights into their behavior. Through a systematic investigation, we aim to expand upon the classical Hermite-Hadamard inequality by incorporating these unique stochastic processes. By employing a recently introduced fractional integral operator, the analysis introduces a novel dimension to the study of mathematical inequalities in the context of stochastic processes.

Keywords: Hermite-Hadamard inequalities; stochastic processes; inequalities in mathematical analysis.

1 Introduction

Stochastic processes, integral for modeling diverse dynamic systems and widely applied across scientific disciplines, possess inherent unpredictability and the capacity to capture complex evolving phenomena. Consequently, they have become indispensable tools for comprehending and simulating real-world dynamics. In recent years, a growing interest has emerged in exploring the convexity properties exhibited by the derivatives of stochastic processes, as these properties play a crucial role in characterizing the behavior and evolution of such processes.

A significant milestone in this field was achieved by Nikodem in 1980, who introduced the concept of convex stochastic processes, thereby sparking an extensive investigation into their regularity properties [13]. Building upon this seminal work, Skowronski [20] made further contributions to our understanding of convex stochastic processes, unveiling results that generalize and expand upon well-established properties of convex functions [20].

Moreover, the Hermite-Hadamard inequality, a key tool for bounding integrals of convex functions, has been expanded to include various forms of convexity and fractional calculus. For instance, it has been adapted to nonlocal conformable fractional integrals, enhancing its applications in complex systems [17]. It has also been applied to n-polynomial exponential-type convexity, which is particularly useful for differential equations [2]. Additionally, Hermite-Hadamard-type inequalities have been linked to the Zipf-Mandelbrot law, illustrating applications in information theory [4]. Fractal-fractional parametric inequalities further extend its relevance in fractal analysis [3]. Lastly, applications of generalized fractional operators offer tools for multiscale engineering problems [5].

Recent studies have showcased the promising potential of extending this inequality to encompass convex stochastic processes [11, 15]. For example, symmetrized stochastic harmonically convexity and harmonically convex stochastic processes have yielded new Hermite-Hadamard type inequalities, expanding their applications in stochastic contexts [12, 14]. Additionally, symmetrized convexity and strongly convex processes provide refined bounds, enhancing understanding of convex behavior in stochastic frameworks [10, 7]. Finally, s-convex stochastic processes reveal unique Hermite-Hadamard inequalities, further advancing this field [18].

1.1 Objectives of the study

This article embarks on exploring Hermite-Hadamard type inequalities tailored specifically for stochastic processes [9]. Through a comprehensive study, our objective is to broaden the scope of the classical Hermite-Hadamard inequality for stochastic processes, as presented in the following theorem:

Theorem 1.1. *For any $(\mu,\nu)\in\mathcal{I}^2$, if $\mathcal{S}_p:\mathcal{I}\times\mathcal{E}\to\mathbb{R}$ is both Jensen-convex and MS-C on $\mathcal{I}$, then the following statement is true:*

$$\mathcal{S}_p\left(\frac{\mu+\nu}{2},\cdot\right)\le\frac{1}{\nu-\mu}\int_\mu^\nu\mathcal{S}_p(\chi,\cdot)d\chi\le\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2},\tag{1}$$

(see [16]).

1.2 Preliminary concepts

1.2.1 Stochastic processes

Consider an arbitrary probability space $(\mathcal{E}, \mathcal{T}, \mathrm{P})$. A function $\Re_v : \mathcal{E} \to \mathbb{R}$ is designated a random variable if it is $\mathcal{T}$-measurable. Extending this concept, a function $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$, where $\mathcal{I} \subset \mathbb{R}$ denotes an interval, is referred to as a stochastic process. For each $\tau \in \mathcal{I}$, the function $\mathcal{S}_p(\tau, \cdot)$ is thereby a random variable.

A stochastic process, denoted as $\mathcal{S}_p(\tau, \cdot) : \tau \in \mathcal{I}$, represents a collection of random variables parameterized by a common probability space $(\mathcal{E}, \mathcal{T}, \mathrm{P})$, where τ is interpreted as time. The notation $\mathcal{S}_p(\tau, \cdot)$, or equivalently $\mathcal{S}_p(\tau, \omega)$ for $\omega \in \mathcal{E}$, signifies the state or position of the process at time τ.

For a specific outcome ω within the sample space $\mathcal{E}$, the mapping $\tau \to \mathcal{S}_p(\tau, \omega)$ characterizes a realization, trajectory, or sample path of the process. When τ is held fixed within the interval $\mathcal{I}$, the mapping depends solely on ω, rendering it a random variable. It follows that $\mathcal{S}_p(\tau, \omega)$ undergoes random variations over time.

In this context, we narrow our focus to continuous-time stochastic processes, wherein the index set is defined as $\mathcal{I} = [0, \infty]$.

Before delving deeper into the properties and characteristics of these processes, let's begin by establishing clear and fundamental definitions that will serve as the foundation for our study.

Definition 1.1. [19] $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ *is recognized as:*

1. *Stochastically continuous over $\mathcal{I}$, ensuring that for each $\chi_0 \in \mathcal{I}$:*
$$\mathrm{P} - \lim_{\chi \to \chi_0} \mathcal{S}_p(\chi, \cdot) = \mathcal{S}_p(\chi_0, \cdot), \tag{2}$$
$\mathrm{P} - \lim$ *being the limit in probability,* [19].

2. *Mean-square continuous (MS-C) throughout $\mathcal{I}$, provided that for every $\chi_0 \in \mathcal{I}$:*
$$\lim_{\chi \to \chi_0} \mathrm{E}\left[\left(\mathcal{S}_p(\chi, \cdot) - \mathcal{S}_p(\chi_0, \cdot)\right)^2\right] = 0, \tag{3}$$
$\mathrm{E}[\mathcal{S}_p(x, \cdot)]$ *denote the expected outcome of the random variable $\mathcal{S}_p(\chi, \cdot)$.*

3. *Mean-square differentiable (MS-D) at a point $\chi \in \mathcal{I}$ is ensured when the derivative $\mathcal{S}'_p : \mathcal{E} \to \mathbb{R}$ is well-defined as follows:*
$$\forall \chi_0 \in \mathcal{I} : \lim_{\chi \to \chi_0} E\left[\frac{\mathcal{S}_p(\chi, \cdot) - \mathcal{S}_p(\chi_0, \cdot)}{\chi - \chi_0} - \mathcal{S}'_p(\chi, \cdot)\right]^2 = 0. \tag{4}$$

Definition 1.2. [1] *Consider a stochastic process $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ where $\mathrm{E}\left[\mathcal{S}_p(\chi; \cdot)^2\right]$ is finite for all $\chi \in \mathcal{I}$.*

Let $\chi_0 = \mu$, $\chi_n = \nu$ and $\chi_{k-1} < \chi_k$, for all $k \in \{1, \dots, n\}$ be a partition of the interval $[\mu; \nu] \subset \mathcal{I}$ such that $\varsigma_k \in [\chi_{k-1}; \chi_k]$.

A random variable $\Re_v : \mathcal{E} \to \mathbb{R}$ is termed mean-square integral (MS-I) of the process $\mathcal{S}_p$ over $[\mu;\nu]$ if the following condition holds:

$$\lim_{n\to\infty} \mathrm{E}\left[\left(\sum_{k=1}^{k=n} \mathcal{S}_p\left(\varsigma_k,\cdot\right)\left(\tau_k-\tau_{k-1}\right)-\Re_v(\cdot)\right)^2\right]=0, \tag{5}$$

for every normal sequence of partitions of $[\mu;\nu]$ and all $\varsigma_k \in [\chi_{k-1};\chi_k]$ for $k=1,\dots,n$. Furthermore, we define:

$$\Re_v(\cdot)=\int_\mu^\nu \mathcal{S}_p(\chi,\cdot)d\chi. \tag{6}$$

These preliminary concepts lay the foundation for our exploration of the Hermite-Hadamard type inequalities for convex stochastic processes.

1.2.2 Definitions

We recall a number of definitions for convex stochastic processes.

Definition 1.3. [13] *A stochastic process $\mathcal{S}_p : \mathcal{I} \subset]0,\infty[\times \mathcal{E} \to \mathbb{R}$ is convex if the inequality for all $\mu,\nu \in \mathcal{I} \subset]0,\infty[$, $p \in \mathbb{R}$ and $\lambda \in [0,1]$, holds:*

$$\mathcal{S}_p\Big(\big[\lambda\mu+(1-\lambda)\nu\big],\cdot\Big) \le \lambda\mathcal{S}_p(\mu,\cdot)+(1-\lambda)\mathcal{S}_p(\nu,\cdot). \tag{7}$$

Definition 1.4. [6] *A stochastic process $\mathcal{S}_p : \mathcal{I} \subset]0,\infty[\times \mathcal{E} \to \mathbb{R}$ is termed p-convex if the following inequality holds almost everywhere for all $\mu,\nu \in \mathcal{I} \subset]0,\infty[$, $p \in \mathbb{R}$, and $\lambda \in [0,1]$,*

$$\mathcal{S}_p\Big(\big[\lambda\mu^p+(1-\lambda)\nu^p\big]^{\frac{1}{p}},\cdot\Big) \le \lambda\mathcal{S}_p(\mu,\cdot)+(1-\lambda)\mathcal{S}_p(\nu,\cdot). \tag{8}$$

Remark 1.1. *The convex stochastic process is a particular case of p-convex for $p=1$.*

Definition 1.5. [8] *A stochastic process $\mathcal{S}_p : \mathcal{I} \subset]0,\infty[\times \mathcal{E} \to \mathbb{R}_+$ is termed geometric-convex if the inequality holds for all $\mu,\nu \in \mathcal{I}$ and $\tau \in [0,1]$,*

$$\mathcal{S}_p\left(\mu^\tau\nu^{1-\tau},\cdot\right) \le \left[\mathcal{S}_p(\mu,\cdot)\right]^\tau\left[\mathcal{S}_p(\nu,\cdot)\right]^{1-\tau}. \tag{9}$$

Definition 1.6. [8] *Let m be a constant in the range $]0,1]$, and $\mathcal{S}_p(\mu,\cdot)$ be a positive stochastic process on the interval $[0,\zeta]$. $\mathcal{S}_p(\mu,\cdot)$ is termed m-geometric-convex on the interval $[0,\zeta]$, if the inequality holds for all $\mu,\nu \in [0,\zeta]$ and $\tau \in [0,1]$,*

$$\mathcal{S}_p\left(\mu^\tau\nu^{m(1-\tau)},\cdot\right) \le \left[\mathcal{S}_p(\mu,\cdot)\right]^\tau\left[\mathcal{S}_p(\nu,\cdot)\right]^{m(1-\tau)}. \tag{10}$$

Definition 1.7. *Let (α,m) be a pair of constants within the range $]0,1]^2$, and $\mathcal{S}_p(\mu,\cdot)$ be a positive stochastic process on the interval $[0,\zeta]$. $\mathcal{S}_p(\mu,\cdot)$ is (α,m)-geometric-convex over the interval $[0,\zeta]$. If the inequality is verified for all $\mu,\nu \in [0,\zeta]$ and $\tau \in [0,1]$,*

$$\mathcal{S}_p\left(\mu^\tau\nu^{m(1-\tau)},\cdot\right) \le \left[\mathcal{S}_p(\mu,\cdot)\right]^{\tau^\alpha}\left[\mathcal{S}_p(\nu,\cdot)\right]^{m(1-\tau^\alpha)}. \tag{11}$$

Definition 1.8. [6] *For a stochastic process $\mathcal{S}_p : \mathcal{I} \subset [0,\infty] \times \mathcal{E} \to \mathbb{R}_+$, where $\mathcal{I}$ is an interval, it is termed s-geometric-convex for some $s \in [0,1]$ if the inequality holds for all $\mu,\nu \in \mathcal{I}$ and $\tau \in [0,1]$,*

$$\mathcal{S}_p\left(\mu^\tau\nu^{1-\tau},\cdot\right) \le \left[\mathcal{S}_p(\mu,\cdot)\right]^{t^s}\left[\mathcal{S}_p(\nu,\cdot)\right]^{(1-\tau)^s}. \tag{12}$$

2 Main Results

For any function $\varphi : \mathbb{R}_0 \to \mathbb{R}_+$ and for given $\mu, \nu, \xi \in \mathbb{R}_+$ and $\alpha, m \in [0,1]$, we establish the following:

$$\psi_{\mu,\nu} = \frac{\psi(\mu)}{\left[\psi\left(\nu^{\frac{1}{m}}\right)\right]^{m}}, \tag{13}$$

$$\zeta(\alpha;\xi) = \begin{cases} 0, & 0 < \xi \leq 1, \\ 1-\alpha, & \xi \geq 1, \end{cases} \tag{14}$$

$$\Delta(\mu,\nu) = \begin{cases} \dfrac{\nu-\mu}{\ln\nu - \ln\mu}, & \mu \neq \nu, \\ \mu, & \mu = \nu, \end{cases} \tag{15}$$

and

$$\Phi(\mu,\nu) = \begin{cases} \dfrac{2\sqrt{\nu}}{\ln\nu - \ln\mu}[\Delta(\sqrt{\mu},\sqrt{\nu}) - \sqrt{\mu}], & \mu \neq \nu, \\ \dfrac{\mu}{2}, & \mu = \nu. \end{cases} \tag{16}$$

Lemma 2.1. *Consider $\mathcal{S}_p : \mathcal{I} \subseteq \mathbb{R}_0 \times \mathcal{E} \to \mathbb{R}$ as a stochastic process that is mean-square differentiable (MS-D) over $\mathcal{I}^\circ$ (the interior of $\mathcal{I}$) , and let $\mu, \nu \in \mathcal{I}^\circ$ with $0 < \mu < \nu$. If $\mathcal{S}_p'$ is mean-square integrable (MS-I) on $[\mu,\nu]$, then,*

$$\begin{aligned} &\frac{1}{\ln(\nu)-\ln(\mu)} \int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot) \\ &\quad = \frac{\big(\ln(\nu)-\ln(\mu)\big)}{4} \int_0^1 \tau\left[\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}}\mathcal{S}_p'\Big(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot\Big) - \mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}}\mathcal{S}_p'\Big(\mu^{1-\tau/2}\nu^{\frac{\tau}{2}},\cdot\Big)\right]\mathrm{d}\tau. \end{aligned} \tag{17}$$

Proof. By implementing the variable transformation $\chi = \mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}}$ with $0 \leq \tau \leq 1$, into the equation $(*)$, we obtain:

$$\begin{aligned} &\frac{\ln(\nu)-\ln(\mu)}{2}\int_0^1 \tau\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}}\mathcal{S}_p'\big(\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}},\cdot\big)\mathrm{d}\tau \\ &\quad = \int_0^1 \tau\mathcal{S}_p'\big(\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}},\cdot\big)\,\mathrm{d}\tau, \\ &\quad = \Big[\tau\mathcal{S}_p\big(\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}},\cdot\big)\Big]_0^1 - \int_0^1 \mathcal{S}_p\big(\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}},\cdot\big)\mathrm{d}\tau \quad (*), \\ &\quad = \mathcal{S}_p\big(\sqrt{\mu\nu},\cdot\big) - \frac{2}{\ln(\nu)-\ln(\mu)}\int_\mu^{\sqrt{\mu\nu}} \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi. \end{aligned}$$

Similarly if $\chi = \mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}}$ for $\tau \in [0,1]$ in $(**)$, we get:

$$\begin{aligned}
\frac{\ln(\nu)-\ln(\mu)}{2}&\int_0^1 \tau\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}}\mathcal{S}_p'\big(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot\big)\mathrm{d}\tau\\
&= -\int_0^1 \tau\Big[\mathcal{S}_p'\big(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot\big)\Big]\,\mathrm{d}\tau,\\
&= -\Big[\tau\mathcal{S}_p\big(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot\big)\Big]_0^1 + \int_0^1 \mathcal{S}_p\big(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot\big)\mathrm{d}\tau \quad (**),\\
&= -\mathcal{S}_p\big(\sqrt{\mu\nu},\cdot\big) + \frac{2}{\ln(\nu)-\ln(\mu)}\int_{\sqrt{ab}}^{b}\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi.
\end{aligned}$$

Thus, Lemma 2.1 is successfully established. □

A straightforward calculation utilizing the preceding lemma yields the following results.

Lemma 2.2. *If $\mu,\nu>0$ and $\mu\neq\nu$, then the following equalities are true:*

$$\int_0^1 \tau\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}}\,\mathrm{d}\tau = \frac{2\sqrt{\nu}}{\ln\nu-\ln\mu}\Big[\Delta\big(\sqrt{\mu},\sqrt{\nu}\big)-\sqrt{\mu}\Big], \tag{18}$$

and

$$\int_0^1 \tau\Big(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}}+\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}}\Big)\mathrm{d}\tau = \Big[\Delta\big(\sqrt{\mu},\sqrt{\nu}\big)\Big]^2, \tag{19}$$

$\Delta(\mu,\nu)$ being defined in (15).

Using these results, we can derive certain Hermite-Hadamard type inequalities for stochastic processes whose derivatives exhibit geometric convexity, m- and (α,m)-geometric convexities, as well as s-geometric convexity.

Theorem 2.1. *$\mathcal{S}_p:\mathcal{I}\subseteq\mathbb{R}_0\times\mathcal{E}\to\mathbb{R}$ is a MS-D stochastic process on $\mathcal{I}^\circ$, and $\mu,\nu\in\mathcal{I}^\circ$ with $0<\mu<\nu$.*

For all $\alpha,m\in]0,1]$, if $\mathcal{S}_p'$ is MS-I on $[\mu,\nu]$, then we have:

$$\begin{aligned}
&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\
&\qquad\leq \frac{\ln(\nu)-\ln(\mu)}{4}\left\{\Delta\left(\Big[\mu\big|\mathcal{S}_p'(\mu,\cdot)\big|\Big]^{\frac{1}{2}},\Big[\nu\big|\mathcal{S}_p'(\nu,\cdot)\big|\Big]^{\frac{1}{2}}\right)\right\}^2,
\end{aligned} \tag{20}$$

with $\Delta(\mu,\nu)$ being defined in (15).

Proof. By utilizing Lemma 2.1, the fact that $\big|\mathcal{S}_p'(\chi,\cdot)\big|$ is (α,m)-geometric convex on $[\mu,\nu]$, and

Lemma 2.2, we get:

$$\begin{aligned}
&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\
&\quad\le \frac{\ln(\nu)-\ln(\mu)}{4}\int_0^1 \tau\left[\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}}\left|\mathcal{S}_p'(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot)\right| + \mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}}\left|\mathcal{S}_p'(\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}},\cdot)\right|\right]\mathrm{d}\tau\\
&\quad\le \frac{\ln(\nu)-\ln(\mu)}{4}\int_0^1 \tau\left\{\Big[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|\Big]^{\frac{\tau}{2}}\Big[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|\Big]^{1-\frac{\tau}{2}} + \Big[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|\Big]^{1-\frac{\tau}{2}}\Big[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|\Big]^{\frac{\tau}{2}}\right\}\mathrm{d}\tau\\
&\quad= \frac{\ln(\nu)-\ln(\mu)}{4}\left\{\Delta\left(\Big[\mu|\mathcal{S}_p'(\mu,\cdot)|\Big]^{\frac{1}{2}},\Big[\nu|\mathcal{S}_p'(\nu,\cdot)|\Big]^{\frac{1}{2}}\right)\right\}^2.
\end{aligned}$$

With this, we complete the proof of Theorem 2.1. □

Theorem 2.2. *Let $\mathcal{S}_p:\mathbb{R}_0\times\mathcal{E}\to\mathbb{R}$ be a MS-D stochastic process on $\mathbb{R}_0$, $(\alpha,m)\in]0,1]^2$, and $\mathcal{S}_p'$ is MS-I on $[\mu,\nu]$ for $\mu,\nu\in\mathbb{R}_0$ with $0<\mu<\nu$. If $\left|\mathcal{S}_p'(\chi,\cdot)\right|$ is (α,m) geometric-convex on $\left[0,\max\left\{\nu,\nu^{\frac{1}{m}}\right\}\right]$, then,*

$$\begin{aligned}
&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\
&\quad\le \frac{\ln(\nu)-\ln(\mu)}{4}\left\{\nu\left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m \left|\mathcal{S}_p'\right|_{\mu,\nu}^{\zeta\left(\alpha;\,\mathcal{S}_p'|_{\mu,\nu}\right)}\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\mu,\nu}^{\alpha}\right)\right.\\
&\qquad\left.+\,\mu\left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m \left|\mathcal{S}_p'\right|_{\nu,\mu}^{\zeta\left(\alpha;|\mathcal{S}_p'|_{\nu,\mu}\right)}\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\nu,\mu}^{\alpha}\right)\right\},
\end{aligned}\tag{21}$$

where $\left|\mathcal{S}_p'\right|_{\mu,\nu}$, $\zeta(\alpha;\xi)$ and $\Phi(\mu,\nu)$ are previously defined in (14) and (16).

Proof. Using the fact that $\mathcal{S}_p'$ is (α,m)-geometrically convex on the interval $\left[0,\max\{\nu,\nu^{\frac{1}{m}}\}\right]$, we obtain the following:

$$\begin{aligned}
\left|\mathcal{S}_p'\left(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot\right)\right| &\le \left|\mathcal{S}_p'(\mu,\cdot)\right|^{\left(\frac{\tau}{2}\right)^\alpha}\left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^{m\left[1-\left(\frac{\tau}{2}\right)^\alpha\right]}\\
&= \left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|_{\mu,\nu}^{\left(\frac{\tau}{2}\right)^\alpha} \le \left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|_{\mu,\nu}^{\zeta\left(\alpha;|\mathcal{S}_p'|_{\mu,\nu}\right)+\frac{\alpha\tau}{2}},
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
\left|\mathcal{S}_p'\left(\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}},\cdot\right)\right| &\le \left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^{m\left[1-\left(\frac{\tau}{2}\right)^\alpha\right]}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\left(\frac{\tau}{2}\right)^\alpha}\\
&= \left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|_{\nu,\mu}^{\left(\frac{\tau}{2}\right)^\alpha} \le \left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|_{\nu,\mu}^{\zeta\left(\alpha;|\mathcal{S}_p'|_{\nu,\mu}\right)+\frac{\alpha\tau}{2}},
\end{aligned}\tag{23}$$

for all $\tau\in[0,1]$.

Using the results of Lemma 2.1, Lemma 2.2 , we get:

$$\begin{aligned}
&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_{\mu}^{\nu}\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi-\mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\
&\quad\le\frac{\ln(\nu)-\ln(\mu)}{4}\times\int_0^1\tau[\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}}\left|\mathcal{S}_p'\left(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot\right)\right|+\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}}\left|\mathcal{S}_p'\left(\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}},\cdot\right)\right|]\mathrm{d}\tau\\
&\quad\le\frac{\ln(\nu)-\ln(\mu)}{4}\int_0^1\tau\left\{\nu\left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|_{\mu,\nu}^{\zeta\left(\alpha;\left|\mathcal{S}_p'\right|_{\mu,\nu}\right)}\left(\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\mu,\nu}^{\alpha}\right)^{\frac{\tau}{2}}\right.\\
&\qquad\left.+\mu\left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|_{\nu,\mu}^{\zeta\left(\alpha;\left|\mathcal{S}_p'\right|_{\nu,\mu}\right)}\left(\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\nu,\mu}^{\alpha}\right)^{\frac{\tau}{2}}\right\}\mathrm{d}\tau\\
&\quad=\frac{\ln(\nu)-\ln(\mu)}{4}\left\{\nu\left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|_{\mu,\nu}^{\zeta\left(\alpha;\left|\mathcal{S}_p'\right|_{\mu,\nu}\right)}\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\mu,\nu}^{\alpha}\right)\right.\\
&\qquad\left.+\mu\left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|_{\nu,\mu}^{\zeta\left(\alpha;\left|\mathcal{S}_p'\right|_{\nu,\mu}\right)}\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\nu,\mu}^{\alpha}\right)\right\},
\end{aligned}$$

which conclude the demonstration of Theorem 2.2. □

Corollary 2.1. *Considering the given conditions in Theorem 2.2.*

1. *For $\alpha = 1$, we get:*

$$\begin{aligned}
&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_{\mu}^{\nu}\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi-\mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\
&\quad\le\frac{\ln(\nu)-\ln(\mu)}{4}\left[\nu\left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\mu,\nu}\right)\right.\\
&\qquad\left.+\mu\left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\nu,\mu}\right)\right].
\end{aligned}\tag{24}$$

2. *For $m = \alpha = 1$, we get:*

$$\begin{aligned}
&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_{\mu}^{\nu}\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi-\mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\
&\quad\le\frac{\ln(\nu)-\ln(\mu)}{4}\left\{\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|\left|\mathcal{S}_p'\right|_{\mu,\nu}^{\zeta\left(\alpha;\left|\mathcal{S}_p'\right|_{\mu,\nu}\right)}\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\mu,\nu}^{\alpha}\right)\right.\\
&\qquad\left.+\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|\left|\mathcal{S}_p'\right|_{\nu,\mu}^{\zeta\left(\alpha;\left|\mathcal{S}_p'\right|_{\nu,\mu}\right)}\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|_{\nu,\mu}^{\alpha}\right)\right\}.
\end{aligned}\tag{25}$$

3. *In the case where $\alpha = m = 1$, we get:*

$$\begin{aligned}
&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_{\mu}^{\nu}\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi-\mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\
&\quad\le\frac{\ln(\nu)-\ln(\mu)}{4}\left\{\Delta\left(\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|\right]^{\frac{1}{2}},\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|\right]^{\frac{1}{2}}\right)\right\}^2.
\end{aligned}\tag{26}$$

Corollary 2.2. *Considering the given conditions in Theorem 2.2,*

1. *Provided that:* $\left|\mathcal{S}_p'(\mu,\cdot)\right| \leq \left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m$ *and* $\left|\mathcal{S}_p'(\nu,\cdot)\right| \leq \left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m$*, we get:*

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\ &\quad\leq \frac{\ln(\nu)-\ln(\mu)}{4}\left\{\nu\left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m \Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|^{\alpha}_{\mu,\nu}\right)+\mu\left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|^{\alpha}_{\nu,\mu}\right)\right\}.\end{aligned}$$

2. *Assuming that:* $\left|\mathcal{S}_p'(\mu,\cdot)\right| \leq \left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},.\right)\right|^m$ *and* $\left|\mathcal{S}_p'(\nu,\cdot)\right| \geq \left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m$*, we get:*

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\ &\quad\leq \frac{\ln(\nu)-\ln(\mu)}{4}\left[\nu\left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m \Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|^{\alpha}_{\mu,\nu}\right)\right.\\ &\qquad\left.+\mu\left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|^{1-\alpha}_{\nu,\mu}\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|^{\alpha}_{\nu,\mu}\right)\right].\end{aligned} \tag{27}$$

3. *Given that:* $\left|\mathcal{S}_p'(\mu,\cdot)\right| \geq \left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m$ *and* $\left|\mathcal{S}_p'(\nu,\cdot)\right| \leq \left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m$*, we get:*

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\ &\quad\leq \frac{\ln(\nu)-\ln(\mu)}{4}\left[\nu\left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|^{1-\alpha}_{\mu,\nu} \Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|^{\alpha}_{\mu,\nu}\right)\right.\\ &\qquad\left.+\mu\left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|^{\alpha}_{\nu,\mu}\right)\right].\end{aligned} \tag{28}$$

4. *In the case where:* $\left|\mathcal{S}_p'(\mu,\cdot)\right| \geq \left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m$ *and* $\left|\mathcal{S}_p'(\nu,\cdot)\right| \geq \left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m$*, we get:*

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\ &\quad\leq \frac{\ln(\nu)-\ln(\mu)}{4}\left[\nu\left|\mathcal{S}_p'\left(\nu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|^{1-\alpha}_{\mu,\nu} \Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|^{\alpha}_{\mu,\nu}\right)\right.\\ &\qquad\left.+\mu\left|\mathcal{S}_p'\left(\mu^{\frac{1}{m}},\cdot\right)\right|^m\left|\mathcal{S}_p'\right|^{1-\alpha}_{\nu,\mu}\Phi\left(1,\frac{\nu}{\mu}\left|\mathcal{S}_p'\right|^{\alpha}_{\nu,\mu}\right)\right].\end{aligned} \tag{29}$$

Theorem 2.3. *Consider the stochastic process* $\mathcal{S}_p : \mathcal{I} \subseteq \mathbb{R}_0 \times \mathcal{E} \to \mathbb{R}$ *as an MS-D process on* $\mathcal{I}^\circ$*, where* $\mathcal{S}_p'$ *is MS-I on* $[\mu,\nu]$ *for* $\mu,\nu \in \mathcal{I}^\circ$ *with* $0<\mu<\nu$*. If* $\left|\mathcal{S}_p'(\chi,\cdot)\right|$ *is s-geometric-convex on* $[\mu,\nu]$ *for some* $s \in (0,1]$*, we get:*

$$\begin{aligned}&\left|\frac{1}{(\ln(\nu)-\ln(\mu))}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\ &\quad\leq \frac{(\ln(\nu)-\ln(\mu))}{4}\left|\mathcal{S}_p'(\mu,\cdot)\right|^{\zeta\left(s;\left|\mathcal{S}_p'(\mu,\cdot)\right|\right)}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\zeta\left(s;\left|\mathcal{S}_p'(\nu,\cdot)\right|\right)}\times\\ &\qquad\left\{\Delta\left(\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{\frac{1}{2}},\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{\frac{1}{2}}\right)\right\}^2,\end{aligned} \tag{30}$$

where $\zeta(\alpha;\xi)$ *and* $\Delta(\mu,\nu)$ *are respectively defined as in* (14) *and* (15)*.*

Proof. Utilizing the fact that $\left|\mathcal{S}_p'(\chi,\cdot)\right|$ is s-geometric convex over the interval $[\mu,\nu]$, we have

$$\begin{aligned}\left|\mathcal{S}_p'\left(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot\right)\right| &\le \left|\mathcal{S}_p'(\mu,\cdot)\right|^{\left(\frac{\tau}{2}\right)^s}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\left(1-\frac{\tau}{2}\right)^s}\\ &\le \left|\mathcal{S}_p'(\mu,\cdot)\right|^{\beta\left(s;\left|\mathcal{S}_p'(\mu,\cdot)\right|\right)+\frac{\alpha\tau}{2}}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\beta\left(s;\left|\mathcal{S}_p'(\nu,\cdot)\right|\right)+\alpha\left(1-\frac{\tau}{2}\right)}\\ &= \left|\mathcal{S}_p'(\mu,\cdot)\right|^{\beta\left(s;\left|\mathcal{S}_p'(\mu,\cdot)\right|\right)}\left[\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{\frac{\tau}{2}}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\beta\left(s;\left|\mathcal{S}_p'(\nu,\cdot)\right|\right)}\left[\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{1-\frac{\tau}{2}},\end{aligned} \tag{31}$$

and

$$\begin{aligned}\left|\mathcal{S}_p'\left(\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}},\cdot\right)\right| &\le \left|\mathcal{S}_p'(\mu,\cdot)\right|^{\left(1-\frac{\tau}{2}\right)^s}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\left(\frac{\tau}{2}\right)^s}\\ &\le \left|\mathcal{S}_p'(\mu,\cdot)\right|^{\beta\left(s;\left|\mathcal{S}_p'(\mu,\cdot)\right|\right)+\alpha\left(1-\frac{\tau}{2}\right)}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\beta\left(s;\left|\mathcal{S}_p'(\nu,\cdot)\right|\right)+\frac{\alpha\tau}{2}}\\ &= \left|\mathcal{S}_p'(\mu,\cdot)\right|^{\beta\left(s;\left|\mathcal{S}_p'(\mu,\cdot)\right|\right)}\left[\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{1-\frac{\tau}{2}}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\beta\left(s;\left|\mathcal{S}_p'(\nu,\cdot)\right|\right)}\left[\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{\frac{\tau}{2}},\end{aligned} \tag{32}$$

with $\tau\in[0,1]$.

By employing the results of Lemmas 2.1 and 2.2 as well as the inequalities (31) and (32) we get:

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi-\mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\ &\le\frac{\ln(\nu)-\ln(\mu)}{4}\int_0^1\tau\left[\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}}\left|\mathcal{S}_p'\left(\mu^{\frac{\tau}{2}}\nu^{1-\frac{\tau}{2}},\cdot\right)\right|+\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}}\left|\mathcal{S}_p'\left(\mu^{1-\frac{\tau}{2}}\nu^{\frac{\tau}{2}},\cdot\right)\right|\right]\mathrm{d}\tau\\ &\le\frac{\ln(\nu)-\ln(\mu)}{4}\left|\mathcal{S}_p'(\mu,\cdot)\right|^{\zeta\left(s;\left|\mathcal{S}_p'(\mu,\cdot)\right|\right)}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\zeta\left(s;\left|\mathcal{S}_p'(\nu,\cdot)\right|\right)}\\ &\quad\times\int_0^1\tau\left\{\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{\frac{\tau}{2}}\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{1-\frac{\tau}{2}}+\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{1-\frac{\tau}{2}}\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{\frac{\tau}{2}}\right\}\mathrm{d}\tau\\ &=\frac{\ln(\nu)-\ln(\mu)}{4}\left|\mathcal{S}_p'(\mu,\cdot)\right|^{\zeta\left(s;\left|\mathcal{S}_p'(\mu,\cdot)\right|\right)}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{\zeta\left(s;\left|\mathcal{S}_p'(\nu,\cdot)\right|\right)}\\ &\quad\times\left\{\Delta\left(\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{\frac12},\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{\frac12}\right)\right\}^2.\end{aligned} \tag{33}$$

□

Corollary 2.3. *Considering the given conditions in Theorem 2.3, we have:*

1. *If* $\left|\mathcal{S}_p'(\mu,\cdot)\right|\le 1$ *and* $\left|\mathcal{S}_p'(\nu,\cdot)\right|\le 1$, *then,*

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi-\mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\ &\qquad\le\frac{\ln(\nu)-\ln(\mu)}{4}\left\{\Delta\left(\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{1/2},\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{1/2}\right)\right\}^2.\end{aligned} \tag{34}$$

2. *If* $\left|\mathcal{S}_p'(\mu,\cdot)\right|\le 1\le\left|\mathcal{S}_p'(\nu,\cdot)\right|$, *then,*

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi-\mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right|\\ &\qquad\le\frac{\ln(\nu)-\ln(\mu)}{4}\left|\mathcal{S}_p'(\nu,\cdot)\right|^{1-s}\left\{\Delta\left(\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{1/2},\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{1/2}\right)\right\}^2.\end{aligned} \tag{35}$$

3. *If* $\left|\mathcal{S}_p'(\nu,\cdot)\right| \leq 1 \leq \left|\mathcal{S}_p'(\mu,\cdot)\right|$, *then,*

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right| \\ &\qquad \leq \frac{\ln(\nu)-\ln(\mu)}{4}\left|\mathcal{S}_p'(\mu,\cdot)\right|^{1-s}\left\{\Delta\left(\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{1/2},\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{1/2}\right)\right\}^2.\end{aligned} \tag{36}$$

4. *If* $\left|\mathcal{S}_p'(\mu,\cdot)\right| \geq 1$ *and* $\left|\mathcal{S}_p'(\nu,\cdot)\right| \geq 1$, *then,*

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right| \\ &\qquad \leq \frac{\ln(\nu)-\ln(\mu)}{4}\left|\mathcal{S}_p'(\mu,\cdot)\mathcal{S}_p'(\nu,\cdot)\right|^{1-s}\left\{\Delta\left(\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|^s\right]^{1/2},\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|^s\right]^{1/2}\right)\right\}^2.\end{aligned} \tag{37}$$

5. *If* $s=1$, *then,*

$$\begin{aligned}&\left|\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi - \mathcal{S}_p(\sqrt{\mu\nu},\cdot)\right| \\ &\qquad \leq \frac{\ln(\nu)-\ln(\mu)}{4}\left\{\Delta\left(\left[\mu\left|\mathcal{S}_p'(\mu,\cdot)\right|\right]^{1/2},\left[\nu\left|\mathcal{S}_p'(\nu,\cdot)\right|\right]^{1/2}\right)\right\}^2.\end{aligned} \tag{38}$$

Theorem 2.4. *Consider a* (α,m)*-geometric-convex stochastic process* $\mathcal{S}_p:\mathbb{R}_0\times\mathcal{E}\to\mathbb{R}_+$ *on* $\left[0,\max\left\{\nu,\nu^{\frac{1}{m}}\right\}\right]$ *and* $\mathcal{S}_p$ *is MS-I on* $[\mu,\nu]$ *for* $0<\mu<\nu$ *and* $(\alpha,m)\in(0,1]^2$, *then,*

$$\begin{aligned}&\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi \\ &\quad \leq \min\Bigg\{\left[\mathcal{S}_p\left(\mu^{\frac{1}{m}},\cdot\right)\right]^{m(1-\alpha)}\mathcal{S}_{p(\nu,\mu)}^{\zeta\left(\alpha;\mathcal{S}_{p(\nu,\mu)}\right)}\Delta\left(\left[\mathcal{S}_p\left(\mu^{\frac{1}{m}},\cdot\right)\right]^{\alpha m},[\mathcal{S}_p(\nu,\cdot)]^\alpha\right), \\ &\qquad\quad \left[\mathcal{S}_p\left(\nu^{\frac{1}{m}},\cdot\right)\right]^{m(1-\alpha)}\mathcal{S}_{p(\mu,\nu)}^{\zeta\left(\alpha;\mathcal{S}_{p(\mu,\nu)}\right)}\Delta\left([\mathcal{S}_p(\mu,\cdot)]^\alpha,\left[\mathcal{S}_p\left(\nu^{\frac{1}{m}},\cdot\right)\right]^{\alpha m}\right)\Bigg\},\end{aligned} \tag{39}$$

where $\mathcal{S}_{p(\mu,\nu)}$, $\zeta(\alpha;\xi)$, *and* $\Delta(\mu,\nu)$ *are defined in* (13), (14), *and* (15).

Proof. For $\chi=\mu^{1-\tau}\nu^\tau$ for $\tau\in[0,1]$ and using the (α,m)-geometric convexity of $\mathcal{S}_p(\chi,\cdot)$ on $\left[0,\max\left\{\nu,\nu^{\frac{1}{m}}\right\}\right]$ we get:

$$\begin{aligned}\frac{1}{\ln(\nu)-\ln(\mu)}\int_\mu^\nu \frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi &= \int_0^1 \mathcal{S}_p\left(\mu^{1-\tau}\nu^\tau,\cdot\right)\mathrm{d}\tau \\ &\leq \int_0^1\left[\mathcal{S}_p\left(\mu^{\frac{1}{m}},\cdot\right)\right]^{m(1-\tau^\alpha)}[\mathcal{S}_p(\nu,\cdot)]^{\alpha^\alpha}\mathrm{d}\tau \\ &\leq \left[\mathcal{S}_p\left(\mu^{\frac{1}{m}},\cdot\right)\right]^m\int_0^1\mathcal{S}_{p(\nu,\mu)}^{\zeta\left(\alpha;\mathcal{S}_{p(\nu,\mu)}\right)+\alpha\tau}\,\mathrm{d}\tau \\ &= \left[\mathcal{S}_p\left(\mu^{\frac{1}{m}},\cdot\right)\right]^{m(1-\alpha)}\mathcal{S}_{p(\nu,\mu)}^{\zeta\left(\alpha;\mathcal{S}_{p(\nu,\mu)}\right)}\Delta\left(\left[\mathcal{S}_p\left(\mu^{\frac{1}{m}},\cdot\right)\right]^{\alpha m},\left[\mathcal{S}_p(\nu,\cdot)\right]^\alpha\right).\end{aligned}$$

The demonstration of Theorem 2.4 has concluded. □

Corollary 2.4. *Considering the given conditions in Theorem 2.4;*

1. *For $\alpha = 1$, we get:*

$$\frac{1}{\ln(\nu)-\ln(\mu)}\int_{\mu}^{\nu}\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi \leq \min\left\{\Delta\left(\left[\mathcal{S}_p\left(\mu^{\frac{1}{m}},\cdot\right)\right]^{m},\mathcal{S}_p(\nu,\cdot)\right),\Delta\left(\mathcal{S}_p(\mu,\cdot),\left[\mathcal{S}_p\left(\nu^{\frac{1}{m}},\cdot\right)\right]^{m}\right)\right\}. \tag{40}$$

2. *In case if $m = 1$, (42) becomes:*

$$\frac{1}{\ln(\nu)-\ln(\mu)}\int_{\mu}^{\nu}\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi \leq \min\left\{\begin{array}{c}[\mathcal{S}_p(\mu,\cdot)]^{1-\alpha}\mathcal{S}_{p(\nu,\mu)}^{\zeta\left(\alpha;\mathcal{S}_{p(\nu,\mu)}\right)},\\ [\mathcal{S}_p(\nu,\cdot)]^{1-\alpha}\,\mathcal{S}_{p(\mu,\nu)}^{\zeta\left(\alpha;\mathcal{S}_{p(\mu,\nu)}\right)}\end{array}\right\}\Delta\left(\mathcal{S}_p^{\alpha}(\mu,\cdot),\mathcal{S}_p^{\alpha}(\nu,\cdot)\right). \tag{41}$$

3. *Assuming that $\alpha = m = 1$, we get the following result:*

$$\frac{1}{\ln(\nu)-\ln(\mu)}\int_{\mu}^{\nu}\frac{\mathcal{S}_p(\chi,\cdot)}{\chi}\,\mathrm{d}\chi \leq \Delta(\mathcal{S}_p(\mu,\cdot),\mathcal{S}_p(\nu,\cdot)). \tag{42}$$

Theorem 2.5. *Consider $\mathcal{S}_{p1}$ and $\mathcal{S}_{p2}$ as (α, m)-geometric-convex stochastic processes on $\left[0,\max\{0,\nu^{\frac{1}{m}}\}\right]$, if $\mathcal{S}_{p1}$ and $\mathcal{S}_{p2}$ are both mean-square integral (MS-I) on $[\mu,\nu]$, where $(\alpha,m)\in[0,1]^2$ and $0<\mu<\nu$, then:*

$$\begin{aligned}
&\frac{1}{\ln(\nu)-\ln(\mu)}\int_{\mu}^{\nu}\frac{\mathcal{S}_{p1}(\chi,\cdot)\mathcal{S}_{p2}(\chi,\cdot)}{\chi}\,\mathrm{d}\chi\\
&\leq \min\left\{(\mathcal{S}_{p1}\mathcal{S}_{p2})_{\nu,\mu}^{\zeta(\alpha;(\mathcal{S}_{p1}\mathcal{S}_{p2})_{\nu,\mu})}\left[\mathcal{S}_{p1}\left(\mu^{\frac{1}{m}},\cdot\right)\mathcal{S}_{p2}(\mu^{\frac{1}{m}},\cdot)\right]^{m(1-\alpha)}\right.\\
&\quad\times\Delta\left(\left[\mathcal{S}_{p1}\left(\mu^{\frac{1}{m}},\cdot\right)\mathcal{S}_{p2}(\mu^{\frac{1}{m}},\cdot)\right]^{\alpha m},[\mathcal{S}_{p1}(\nu,\cdot)\mathcal{S}_{p2}(\nu,\cdot)]^{\alpha}\right)\\
&\quad(\mathcal{S}_{p1}\mathcal{S}_{p2})_{\mu,\nu}^{\zeta(\alpha;(\mathcal{S}_{p1}\mathcal{S}_{p2})_{\mu,\nu})}\left[\mathcal{S}_{p1}\left(\nu^{\frac{1}{m}},\cdot\right)\mathcal{S}_{p2}(\nu^{\frac{1}{m}},\cdot)\right]^{m(1-\alpha)}\\
&\quad\left.\times\Delta\left([\mathcal{S}_{p1}(\mu,\cdot)\mathcal{S}_{p2}(\mu,\cdot)]^{\alpha},\left[\mathcal{S}_{p1}\left(\nu^{\frac{1}{m}},\cdot\right)\mathcal{S}_{p2}(\nu^{\frac{1}{m}},\cdot)\right]^{\alpha m}\right)\right\},
\end{aligned} \tag{43}$$

where $\mathcal{S}_{p(\mu,\nu)}$, $\zeta(\alpha;\xi)$, and $\Delta(\mu,\nu)$ are respectively defined as in (13), (14), and (15).

Proof. To demonstrate this theorem, we employ a similar methodology to that utilized in proving Theorem 2.4. ☐

Corollary 2.5. *Considering the given conditions in Theorem 2.5,*

1. *For $\alpha = 1$, we get the following result:*

$$\frac{1}{\ln(\nu) - \ln(\mu)} \int_{\mu}^{\nu} \frac{\mathcal{S}_{p1}(\chi, \cdot)\mathcal{S}_{p2}(\chi, \cdot)}{\chi} \,\mathrm{d}\chi \leq \min\left\{ \Delta\left(\left[\mathcal{S}_{p1}\left(\mu^{\frac{1}{m}}, \cdot\right) \mathcal{S}_{p2}(\mu^{\frac{1}{m}}, \cdot) \right]^m, \mathcal{S}_{p1}(\nu, \cdot)\mathcal{S}_{p2}(\nu, \cdot) \right), \Delta\left(\mathcal{S}_{p1}(\mu, \cdot)\mathcal{S}_{p2}(\mu, \cdot), \left[\mathcal{S}_{p1}\left(\nu^{\frac{1}{m}}, \cdot\right) \mathcal{S}_{p2}(\nu^{\frac{1}{m}}, \cdot) \right]^m \right) \right\}. \tag{44}$$

2. *Provided that $m = 1$, (46) becomes:*

$$\frac{1}{\ln(\nu) - \ln(\mu)} \int_{\mu}^{\nu} \frac{\mathcal{S}_{p1}(\chi, \cdot)\mathcal{S}_{p2}(\chi, \cdot)}{\chi} \,\mathrm{d}\chi \leq \min\left\{ \begin{array}{c} (\mathcal{S}_{p1}\mathcal{S}_{p2})_{\nu,\mu}^{\zeta(\alpha;(\mathcal{S}_{p1}\mathcal{S}_{p2})_{\nu,\mu})} \left[\mathcal{S}_{p1}(\mu, \cdot)\mathcal{S}_{p2}(\mu, \cdot) \right]^{1-\alpha}, \\ (\mathcal{S}_{p1}\mathcal{S}_{p2})_{\mu,\nu}^{\zeta(\alpha;(\mathcal{S}_{p1}\mathcal{S}_{p2})_{\mu,\nu})} \left[\mathcal{S}_{p}(\nu, \cdot)\mathcal{S}_{p2}(\nu, \cdot) \right]^{1-\alpha} \end{array} \right\} \Delta\Big(\left[\mathcal{S}_{p1}(\mu, \cdot)\mathcal{S}_{p2}(\mu, \cdot) \right]^{\alpha}, \left[\mathcal{S}_{p1}(\nu, \cdot)\mathcal{S}_{p2}(\nu, \cdot) \right]^{\alpha} \Big).$$

3. *Assuming that $\alpha = m = 1$, we get:*

$$\frac{1}{\ln(\nu) - \ln(\mu)} \int_{\mu}^{\nu} \frac{\mathcal{S}_{p1}(\chi, \cdot)\mathcal{S}_{p2}(\chi, \cdot)}{\chi} \,\mathrm{d}\chi \leq \Delta\Big(\mathcal{S}_{p1}(\mu, \cdot)\mathcal{S}_{p2}(\mu, \cdot), \mathcal{S}_{p1}(\nu, \cdot)\mathcal{S}_{p2}(\nu, \cdot) \Big). \tag{45}$$

Theorem 2.6. *If $\mathcal{S}_p : \mathcal{I} \subseteq \mathbb{R}_0 \times \mathcal{E} \to \mathbb{R}_+$ are s-geometric-convex stochastic processes on $\mathcal{I}^\circ$ for $s \in (0, 1]$, $\mu, \nu \in \mathcal{I}^\circ$ with $0 < \mu < \nu$ and if $\mathcal{S}_p$ is MS-I on $[\mu, \nu]$ the following inequality is valid:*

$$\frac{1}{\ln(\nu) - \ln(\mu)} \int_{\mu}^{\nu} \frac{\mathcal{S}_p(\chi, \cdot)}{\chi} \,\mathrm{d}\chi \leq \left[\mathcal{S}_p(\mu, \cdot) \right]^{\zeta(s, \mathcal{S}_p(\mu, \cdot))} \left[\mathcal{S}_p(\nu, \cdot) \right]^{\zeta(s, \mathcal{S}_p(\nu, \cdot))} \Delta\Big(\mathcal{S}_p^s(\mu, \cdot), \mathcal{S}_p^s(\nu, \cdot) \Big), \tag{46}$$

where $\zeta(\alpha; \xi)$ and $\Delta(\mu, \nu)$ are previously defined in (13), (14), and (15).

.

Proof. If we put $\chi = \mu^{1-\tau}\nu^{\tau}$, for $\tau \in [0, 1]$ and utilizing the s-geometric convexity of $\mathcal{S}_p(\chi, \cdot)$ on $[\mu, \nu]$ give,

$$\frac{1}{\ln(\nu) - \ln(\mu)} \int_{\mu}^{\nu} \frac{\mathcal{S}_p(\chi, \cdot)}{\chi} \,\mathrm{d}\chi = \int_0^1 \mathcal{S}_p\left(\mu^{1-\tau}\nu^{\tau}, \cdot \right) \mathrm{d}\tau \leq \int_0^1 \left[\mathcal{S}_p(\mu, \cdot) \right]^{(1-\tau)^s} \left[\mathcal{S}_p(\nu, \cdot) \right]^s \mathrm{d}\tau \tag{47}$$

$$\leq \int_0^1 \left[\mathcal{S}_p(\mu, \cdot) \right]^{\zeta(s, \mathcal{S}_p(\mu, \cdot)) + s(1-\tau)} \left[\mathcal{S}_p(\nu, \cdot) \right]^{\zeta(s, \mathcal{S}_p(\nu, \cdot)) + st} \mathrm{d}\tau \tag{48}$$

$$= \left[\mathcal{S}_p(\mu, \cdot) \right]^{\zeta(s, \mathcal{S}_p(\mu, \cdot))} \left[\mathcal{S}_p(\nu, \cdot) \right]^{\zeta(s, \mathcal{S}_p(\nu, \cdot))} \Delta\Big(\mathcal{S}_p^s(\mu, \cdot), \mathcal{S}_p^s(\nu, \cdot) \Big). \tag{49}$$

The demonstration of Theorem 2.6 has concluded. □

Theorem 2.7. *Let $\mathcal{S}_{p1}, \mathcal{S}_{p2} : \mathcal{I} \subseteq \mathbb{R}_0 \times \mathcal{E} \to \mathbb{R}_+$ be s-geometric-convex stochastic processes on $\mathcal{I}^\circ$ for $s \in (0,1]$, $\mu, \nu \in \mathcal{I}^\circ$ with $0 < \mu < \nu$ and $\mathcal{S}_{p1}$ and $\mathcal{S}_{p2}$ are MS-I on $[\mu, \nu]$, then,*

$$\frac{1}{\ln(\nu) - \ln(\mu)} \int_\mu^\nu \frac{\mathcal{S}_{p1}(\chi, \cdot)\mathcal{S}_{p2}(\chi, \cdot)}{\chi} \,\mathrm{d}\chi \tag{50}$$

$$\leq \big[\mathcal{S}_{p1}(\mu, \cdot)\mathcal{S}_{p2}(\mu, \cdot)\big]^{\zeta(s, \mathcal{S}_{p1}(\mu,\cdot)\mathcal{S}_{p2}(\mu,\cdot))} \big[\mathcal{S}_{p1}(\nu, \cdot)\mathcal{S}_{p2}(\nu, \cdot)\big]^{\zeta(s, \mathcal{S}_{p1}(\nu,\cdot)\mathcal{S}_{p2}(\nu,\cdot))} \tag{51}$$

$$\times \Delta\Big(\big[\mathcal{S}_{p1}(\mu, \cdot)\mathcal{S}_{p2}(\mu, \cdot)\big]^s, \big[\mathcal{S}_{p1}(\nu, \cdot)\mathcal{S}_{p2}(\nu, \cdot)\big]^s\Big), \tag{52}$$

where $\zeta(\alpha; \xi)$ is defined in (14) and $\Delta(\mu, \nu)$ in (15).

Proof. To demonstrate this theorem, we employ a similar methodology to that utilized in proving Theorem 2.6. □

3 Conclusion

This study has delved into the exploration of novel inequalities of Hermite-Hadamard type specifically tailored for positive convex stochastic processes. Through the utilization of a recently introduced fractional integral operator, we have added a distinctive dimension to the analysis of stochastic processes, thereby contributing significantly to the development of mathematical inequalities in this context. This research opens avenues for further investigations and applications within the realm of stochastic processes with convexity-preserving properties, offering valuable insights into their behavior and potential applications in various fields.

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